Analytic formulae for Commodity Contingent Valuation

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Abstract

Due to the complexity of commodity prices dynamics, valuation of commodity contingent claims is carried out in the extant literature via ad-hoc solutions, which are very complex and sometimes include approximations. In this article we present up to date techniques and ready-to-implement solutions to perform the valuation, using some well known results in stochastic calculus to simplify formulae and deductions and a general algorithm to compute formulae in any dimension. We specifically show how this general framework can be implemented in the context of the two-factor model by Schwartz (1997), obtaining simpler expressions and more precise estimates than the up-biased approximations given by the author. Moreover, we show how to obtain the expression for the futures price given by Schwartz and Smith (2000) in a simpler way, avoiding unnecessary limit steps.

1. Introduction

Itô calculus has become the main approach in derivatives valuation theory since it was first used in Finance (Black and Scholes, 1972). The same methodology was first used in the valuation of commodity contingent claims (see for example Brennan and Schwartz, 1985, Paddock et al., 1988, among others), i.e. by assuming that asset prices follow a geometric Brownian motion, the classical Black-Scholes formulae can be used with slight modifications (if any). Subsequently several authors, such as Laughton and Jacobi (1993) Ross (1997) or Schwartz (1997), have considered that a mean-reverting process is more appropriate to model the stochastic behaviour of commodity prices, pointing out that the geometric Brownian motion hypothesis implies a constant rate of growth in the commodity price and a variance of futures prices increasing monotonically with time, which are not realistic assumptions. The idea behind mean-reverting processes is that the supply of the commodity, by increasing or decreasing, will force its price towards an equilibrium (or long-term mean) price level[1].

In spite of their attractiveness, these one-factor mean-reverting models are not very realistic since they generate a constant volatility term structure of futures returns, instead of a decreasing term structure, as observed in practice. Gibson and Schwartz (1990) and Schwartz (1997) propose a two-factor model, where the second factor is the convenience yield, which is also assumed to follow a mean-reverting process. Schwartz and Smith (2000) propose a two-factor model allowing for mean reversion in short-term prices and uncertainty in the equilibrium (long-term) price to which prices revert, which is equivalent to the Schwartz (1997) one. Schwartz (1997) also considers

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[1] See Schwartz (1997) and Schwartz and Smith (2000) for an excellent discussion of these issues.

a three-factor model, extending the Gibson-Schwartz (1990) model to include stochastic interest rates. Cortazar and Schwartz (2003) propose a three-factor model, which is an extension of the Schwartz (1997) two-factor model, where all three factors are calibrated using only commodity prices. More recently Cortazar and Naranjo (2006) extend two and three factor models to an arbitrary number of factors (N-factor model).

Unfortunately, the application of the standard Black-Scholes valuation framework is not easy in the context of commodity contingent valuation, given the complex dynamics of commodity prices. This is the reason why the studies on commodity contingent valuation usually present very complex ad-hoc solutions and sometimes include approximations or limit steps. In this article we show how to simplify formulae and deductions, computing the explicit, directly implementable general formula, based on well known results in stochastic calculus.

Specifically, after describing the general theoretical model for commodity contingent valuation, we present two specific applications. Firstly, we show how this general framework can be implemented in the context of the two-factor model by Schwartz (1997), obtaining simpler expressions and more precise estimates than the approximations given by the author. It is also shown that the approximations by Schwartz tend to overestimate the parameters, a fact that, as we will see, becomes important in the valuation of commodity contingent claims. Secondly, we shall show how to obtain the expression for the futures price and volatility of futures returns given by Schwartz (1997) and Schwartz and Smith (2000) in a simpler way, avoiding unnecessary partial differential equations or limit steps.

This paper is organized as follows. The general methodology for commodity contingent valuation and volatility estimation is presented in Section 2. Section 3 describes how these formulae can be used in practice and proposes a ready-to-implement algorithm to estimate any linear model which is evaluated in terms of computer time. Section 4 shows how to obtain more precise estimators of the parameters in the two-factor model by Schwartz (1997). Section 5 shows how to simplify the deduction of the futures price in the two-factor model by Schwartz and Smith (2000), avoiding unnecessary limit steps. Finally, section 6 concludes with a summary and discussion.

2. Theoretical Model

Contract Valuation

Most of the models proposed in the literature for the stochastic behaviour of commodity prices can be summarized by means of the following system:

$$\begin{cases} dX_t = (b + AX_t)dt + RdW_t \\ Y_t \quad = cX_t \end{cases} \tag{1}$$

where Y_t is the commodity price (or its log), *b*, *A*, *R* and *c* are deterministic matrices[2] independent of *t* ($b \in \Re^n, A, R \in \Re^{n x n}$, $c \in \Re^n$) and W_t is a *n*-dimensional canonical Brownian motion (i.e. all components uncorrelated and its variance equal to unity). Usually, the estimation of these matrices can be simplified, as they can be assumed to depend in a predefined way of some estimable values, called structural parameters or hyperparameters (for example, if *A* is 2x2, instead of computing four values one may

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[2] *R* does not have to be computed, as all formulae shall use *RR'*.

assume, as in Schwartz, 1997, that $A=\begin{pmatrix}0 & -1\\ 0 & -\kappa\end{pmatrix}$ where κ is the hyperparameter to be estimated).

As it shall be proven in appendix B the solution of this problem is:

$$X_t = e^{At}\left[X_0 + \int_0^t e^{-As} b ds + \int_0^t e^{-As} R dW_s\right] \tag{2}$$

We shall assume now that *A* is diagonalizable with $A = PDP^{-1}$ and $D=\begin{pmatrix}0 & 0\\ 0 & D_1\end{pmatrix}$ diagonal[3]. Let us define the auxiliary quantities:

$$J(t) = P\begin{pmatrix}It & 0\\ 0 & D_1^{-1}[\exp(D_1 t) - I]\end{pmatrix}P^{-1} \tag{3}$$

$$G(t) = \exp(At)Pvec^{-1}\left\{\left[\int_0^t \exp(Ds)\otimes\exp(Ds)ds\right]vec\left(P^{-1}RR'P'\right)\right\}\left(P^{-1}\right)'\exp(At)' \tag{4}$$

This integral can be computed explicitly, but depends on the eigenvalues (see appendix A).

Using (2) and the results in Appendix A about integrals, it is evident that, given X_0, X_t is Gaussian, with mean and variance:

$$E[X_t] = e^{At}X_0 + J(t)b \ , \ Var[X_t] = G(t). \tag{5}$$

Which yields that Y_t is also Gaussian with $E[Y_t] = cE[X_t]$, $Var[Y_t] = cVar[X_t]c'$

Under the risk-neutral measure, the dynamics are exactly the same as in (1) but changing *b* into a different b^* which contains the risk premia (all other matrices stay

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[3] To the best of the authors knowledge all models in the existing literature fulfil this restriction, most of them directly by imposing *A* to be diagonal. Notable exceptions where *A* is not diagonal but diagonalizable are the Schwartz (1997) model or the cycles in Harvey (1991).

the same) so, using this measure and conditional to X_0, X_t is Gaussian. To compute the risk-neutral mean and variance of X_t and Y_t we must substitute *b* for b^* in (5), thus providing a valuation scheme for all sorts of commodity contingent claims such as financial derivatives on commodity prices, real options, investment decisions, etc.

If Y_t is the log of the commodity price (S_t), it is easy to prove (just by the properties of the log-normal distribution) that the price of a futures contract traded at time "*t*" with maturity at time "*t+T*" is:

$$F(t,T) = \exp\left(ce^{AT} X_t + cJ(T)b^* + \frac{1}{2} cG(T)c' \right) \tag{6}$$

This methodology is general, feasible for all kind of problems, at least when the parameters in (1) are independent of *t*, and much simpler than the ad-hoc solutions presented in the literature, that can only be used in the concrete problem for which they were developed and need complex procedures such as partial differential equations (Schwartz 1997) or limit steps (Schwartz-Smith 2000). Even more, these formulae can be implemented directly in any mathematical oriented computer language, such as Matlab or C++ *regardless on the size of the matrices or their dependence of the hyperparameters, using the matrices directly as inputs.* So there is no need to compute explicit formulae each time we use a different model. It possible to use the same script (changing the way the matrix depend on the hyperparameters) for any model.

Volatility of Future Returns

We can define the squared volatility of a futures contract traded at time "*t*" with maturity at time "*t+T*" as[4]: $\lim_{h\to 0}\frac{Var\left[\log F_{t+h,T}-\log F_{t,T}\right]}{h}$. In appendix C it is proved that it is the expected value of the square of the coefficient of the Brownian motion (σ_t) in the expansion $d\log\left(F_{t,T}\right)=\mu_t ds+\sigma_t dW_t^F$, where W_t^F is a scalar canonical Brownian motion, as long as μ_t is mean squared bounded in an interval containing t (it does not matter whether it is a function of $F_{t,T}$ or not) and $E\left[\sigma_t^2\right]$ is continuous in t.

In the general problem of this article these conditions are satisfied. Therefore, after taking logarithms and differentials on both sides of Equation (6), we can obtain that:

$$d\left(\log F_{t,T}\right)=ce^{AT}dX_t=ce^{AT}\left[b+AX_t\right]dt+ce^{AT}R\,dW_t$$

So, the squared volatility is simply[5]:

$$ce^{AT}RR'e^{AT}{}'c'. \tag{7}$$

3. Discretization and estimation issues

This section is devoted to provide a practitioner's guide to the use of the above results. Suppose that we observe a forward curve $F\left(t,T\right)$ of *N* futures prices and wish to estimate a linear multifactor model as in (1). First of all, we need a discrete version of (1). Let Δt be the interval of discretization.

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[4] The same results would be obtained if the volatility were defined as: $\lim_{h\to 0}\frac{Var\left[\log F_{t+h,T-h}-\log F_{t,T}\right]}{h}$.

[5] Note again that *R* does not need to be computed as RR' is the noise covariance matrix.

As stated above $E[X_t] = e^{At} X_0 + J(t)b$ and $Var[X_t] = G(t)$. Consequently, it is easy to prove that:

$$\begin{cases} X_{t+\mathrm{At}} & = b_D + A_D X_t + \eta_t \\ y_t & = d + c_d X_t + \varepsilon_t \end{cases} \tag{8}$$

where $y_t = [\log(F(t,T_1)),\ldots,\log(F(t,T_N))]'$ is the log of the full forward curve, $A_D = \exp(A\Delta t)$, $b_D = J(\Delta t)b$, $E[\eta_t] = 0$, $Var(\eta) = G(\Delta t)$,

$$d_i = cJ(T_i)b^* + \frac{1}{2}cG(T_i)c' \quad i = 1 \ldots N \text{ and } c_D = \begin{pmatrix} c\exp(AT_1) \\ \ldots \\ c\exp(AT_N) \end{pmatrix}.$$

Of course, the measurement noise (ε_t) is user-defined. The most usual convention, followed by Schwartz (1997), Schwartz and Smith (2000), Cortazar and Naranjo (2006) among others, is $E[\varepsilon_t] = 0$, $Var[\varepsilon_t] = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \ldots & 0 \\ 0 & 0 & \sigma_N \end{pmatrix}$

The process to estimate a model is as follows:

1. Given a set of hyperparameters ϕ, make explicit the dependence of the continuous time system matrices $A(\phi), c(\phi)$ and so on in (1)
2. Compute the discrete-time system (8). This can be done using the formulae (3) and (4) or directly via the integrals in appendix B. The easiest way is obviously to compute them by hand and insert them in the program. However, the computer can do it, using the formulae (3) and (4) each iteration at a moderate additional computational cost (thus allowing the user to write a single program for all models, instead of changing it each time).

3. Estimate the parameters in the models by a log-likelihood algorithm. See Hamilton (1994) for details on estimating a state-space model.

From the authors' point of view, unless the user always deals with the same kind of model, the increasing complexity of using formulae (3) and (4) in each iteration is a price worth paying by having a single general program.

We would like to stress the importance of formulae (3) and (4). Without them, unless the practitioner writes a separate script for each model, he would have to compute (via a symbolic processor such as Matlab Symbolic Toolbox) an integral in each iteration. The computational cost of that is burdensome, approximately 100 times the one with the formulae, which is two orders of magnitude higher.

To proof this, we have estimated the Schwartz and Smith (2000) and Cortazar and Schwartz (2003) models with different data sets, representative of the kind of series a practitioner is likely to work with. Here, it suffices to say that they are a two factor (Schwartz and Smith, 2000) and a three factor (Cortazar and Schwartz, 2003) model with 8 and 13 identifiable hyperparameters respectively. The data set employed consists on weekly observations of Henry Hub natural gas, WTI crude oil futures prices (both of them traded at NYMEX) and Brent crude oil futures prices (traded at ICE). The data set for Henry Hub natural gas is made of contracts F1, F5, F9, F13, F17, F21, F25, F29, F33, F37, F41, F44 and F48 where F1 is the contract closest to maturity, F2 is the second contract closest to maturity and so on. This data set contains 330 quotations of each contract from 12/03/2001 to 03/24/2008. The data set for WTI crude oil is made of contracts F1, F4, F7, F10, F13, F16, F19, F22, F25 and F28. This data set contains 654 quotations of each contract from 9/18/1995 to 03/24/2008. The data set for Brent crude oil is made of contracts F1, F4, F7, F10, F12, F16-18, F22-24 and F31-36. This data set contains 537 quotations of each contract from 12/15/1997 to 03/24/2008. These data sets have been chosen taking into account that futures contracts with long-term and short-term maturities are necessary to estimate properly the parameters of the long-term and the short-term factors.

In Table 1 a brief summary of the time needed for an evaluation of the log-likelihood function is given, specifying the data and model used (two factors means Schwartz and Smith, 2000, model, three factors means Cortazar and Schwartz, 2003). Note that, as all quantities are given in milliseconds, a 30% less for the formulae (implementing each case separately) is not a big reward. All experiments were made with an x86 Intel Celeron (Family 6 Model 8 Stepping 3, 261.616 Kb RAM).

In order to illustrate this fact, we have also included another Table (number 2) where the estimation time is given for the general case and the estimation for each case separately (using the theoretical formulae for integrals would be too slow). As the reader can see, the difference is small enough and, from the authors' point of view, it is not worth the effort to compute formulae by hand case by case instead of using matrix forms. Note that the difference is estimating a model in a minute and a minute and a half, even with a rather old computer.

3. The precise estimation of the Schwartz (1997) two-factor model

Let us consider the two-factor model in Schwartz (1997). Let S_t and δ_t be the spot price of a commodity and its instantaneous convenience yield at time t. The model can be expressed as:

$$
\begin{aligned}
dS_t &= (\mu - \delta_t) S_t dt + \sigma_1 S_t dz_1 \\
d\delta_t &= \kappa(\alpha - \delta_t) dt + \sigma_2 dz_2
\end{aligned}
$$

The standard Brownian motions, dz_1 and dz_2, are assumed to be correlated, i.e. $dz_1 dz_2 = \rho dt$. The parameter μ is the long-term total return on the commodity, κ is the mean-reverting coefficient, α is the long-term convenience yield, and finally σ_1 and σ_2 are the volatilities of the spot price and the convenience yield respectively.

Defining $Y_t = \ln(S_t)$ and applying Itô's Lemma, the model, under the risk-neutral measure, can be expressed as:

$$
\begin{aligned}
dY_t &= \left(r - \delta_t - \sigma_1^2 / 2\right) dt + \sigma_1 dz_1^* \\
d\delta_t &= \left[\kappa(\alpha - \delta_t) - \lambda\right] dt + \sigma_2 dz_2^*
\end{aligned}
$$

Where $d\,z_1^*$ and $d\,z_2^*$ are the Brownian motions under the equivalent martingale measure, which are assumed to be correlated, i.e. $d\,z_1^* dz_2^* = \rho\, dt$, λ is the market price of risk associated to the convenience yield and r is the risk-free interest rate.

If we define the state vector as $X_t = (Y_t\,, \delta_t)'$ and after applying the results in section 2, it is easy to prove that X_t is normally distributed with a mean and variance given by the following expressions[6,7]:

$$
E^*[X_t] = \begin{pmatrix} (r - \sigma_1^2 / 2 - \alpha)t + \alpha(1 - e^{-kt})\lambda / k \\ \alpha(1 - e^{-kt}) \end{pmatrix} + \begin{pmatrix} 1 & -(1 - e^{-kt}) / k \\ 0 & e^{-kt} \end{pmatrix} X_0
$$

$$
Var^*[X_t] = \begin{pmatrix} \sigma_1 t + 2\sigma_1\sigma_2\rho(1 - e^{-kt} - kt)/k^2 - \sigma_2^2(3 - 4e^{-kt} + e^{-2kt} - 2kt)/2k^3 & \sigma_1\sigma_2\rho(1 - e^{-kt})/k + \sigma_2^2(1 - 2e^{-kt} + e^{-2kt})/2k^2 \\ \sigma_1\sigma_2\rho(1 - e^{-kt})/k + \sigma_2^2(1 - 2e^{-kt} + e^{-2kt})/2k^2 & \sigma_2^2(1 - e^{-kt})/k \end{pmatrix}
$$

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[6] E*[] and Var*[] are the mean and variance under the risk neutral measure.

[7] Here, in this section, we shall use the formulas in integral form, without resorting to (3) and (4).

Therefore, $Y_t = \ln(S_t)$ is also Gaussian, under the risk-neutral measure, with mean:

$$Y_0 - \delta_0(1-e^{-kt})/k + (r - \sigma_Y^2/2 - \alpha^*)t + \alpha^* k(1-e^{-kt})/k^2$$

where $\alpha^* = \alpha - \lambda/\kappa$, and variance:

$$(\sigma_1^2 + \sigma_2^2/k^2 - 2\sigma_1\sigma_2\rho/k)t + (1-e^{-2\kappa t})\sigma_2^2/2\kappa^3 + 2(\rho\sigma_1\sigma_2 - \sigma_2^2/k)\left(1-e^{-\kappa t}\right)/\kappa^2 .$$

Finally, given that the spot price S_t is lognormal, the futures price can be expressed as:

$$\begin{gathered} F_{0,T} = E^*[S_T] = \exp\left(E^*[Y_T] + \frac{1}{2}Var^*[Y_T]\right) = \\ \exp\{Y_0 - \delta_0(1-e^{-kt})/k + (r - \alpha^* + \sigma_2^2/2k^2 - \sigma_1\sigma_2\rho/k)T \\ + (1-e^{-2\kappa t})\sigma_2^2/4\kappa^3 + (\alpha^* k + \rho\sigma_1\sigma_2 - \sigma_2^2/k)\left(1-e^{-\kappa t}\right)/\kappa^2\} \end{gathered}$$

This is the result already obtained in Schwartz (1997), equation 20, but avoiding unnecessary partial differential equations.

Using the results in section 2, the squared volatility of futures returns can be expressed as:

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \left(e^{-\kappa T}-1\right)/\kappa \\ 0 & e^{-\kappa T} \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \left(e^{-\kappa T}-1\right)/\kappa & e^{-\kappa T} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} =$$

$$\sigma_1^2 + \left(1-e^{-\kappa T}\right)^2 \sigma_2^2/\kappa^2 - 2\left(1-e^{-\kappa T}\right)\rho\sigma_1\sigma_2/\kappa$$

which is the same formula as in Schwartz (1997), equation 40.

Now let us express the model in its discrete-time version. Following Schwartz's notation the model can be expressed as[8]:

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[8] Note that these expressions are just the discrete-time counterpart of expressions (8) with $A_D = M_t$ and $d = c_t$ in our notation.

$$X_t = c_t + M_t X_{t-1} + \psi_t$$

where:

$$c_t = \begin{pmatrix} (\mu - \sigma_1^2/2 - \alpha)\Delta t + \alpha(1 - e^{-k\Delta t})\lambda / k \\ \alpha(1 - e^{-k\Delta t}) \end{pmatrix}, M_t = \begin{pmatrix} 1 & \left(e^{-\kappa\Delta t} - 1\right)/\kappa \\ 0 & e^{-\kappa\Delta t} \end{pmatrix} \qquad (9)$$

and the error term vector, denoted as ψ_t, is a n-vector of serially uncorrelated Gaussian disturbances with zero mean and variance given by the following expression:

$$Var[\psi_t] =$$

$$\begin{pmatrix} \sigma_1\Delta t + \frac{2\sigma_1\sigma_2\rho(1-e^{-k\Delta t} - k\Delta t)}{k^2} - \frac{\sigma_2^2(3-4e^{-k\Delta t} + e^{-2k\Delta t} - 2k\Delta t)}{2k^3} & \frac{\sigma_1\sigma_2\rho(1-e^{-k\Delta t})}{k} + \frac{\sigma_2^2(1-2e^{-k\Delta t} + e^{-2k\Delta t})}{2k^2} \\ \frac{\sigma_1\sigma_2\rho(1-e^{-k\Delta t})}{k} + \frac{\sigma_2^2(1-2e^{-k\Delta t} + e^{-2k\Delta t})}{2k^2} & \frac{\sigma_2^2(1-e^{-k\Delta t})}{k} \end{pmatrix} \qquad (10)$$

If we perform a Taylor expansion when Δt tends to zero and drop all terms of order higher than one, we get expressions 35 in Schwartz (1997):

$$c_t = \begin{pmatrix} (\mu - \sigma_1^2/2)\Delta t \\ \alpha k \Delta t \end{pmatrix}, \ M_t = \begin{pmatrix} 1 & -\Delta t \\ 0 & 1 - \kappa\Delta t \end{pmatrix} \text{ and } Var[\psi_t] = \begin{pmatrix} \sigma_1^2\Delta t & \sigma_1\sigma_2\rho\Delta t \\ \sigma_1\sigma_2\rho\Delta t & \sigma_2^2\Delta t \end{pmatrix}$$

Therefore, we can conclude that Schwartz (1997) uses a discrete-time version of the model which is an approximation to the precise one presented above, which is given by expressions (9) and (10). As we will see below, these divergences, specially the more accurate estimator of the variance of the residual, $Var[\psi_t]$, given by expression (10), are important in the valuation of commodity contingent claims.

Next we are going to compare the empirical performance of both estimation procedures, i.e. the precise version of the estimates given in this paper and the approximate version in Schwartz (1997), using the same data set as in Schwartz (1997). Specifically, the data set is composed of weekly observations of NYMEX WTI crude oil futures contracts, with maturity 1, 3, 5, 7, and 9 months, from 1/1/1985 to 02/13/1995. We have a total of 529 observations[9]. WTI futures prices with one month to maturity are depicted in Figure 1.

The results of the estimation of the two factor model by Schwartz obtained with both estimation procedures are contained in Table 3. The main differences between the results obtained with both procedures are found in the values of κ(the mean-reverting parameter), σ_2 (the volatility of the convenience yield) and λ (the market price of risk associated to the convenience yield). Specifically, the value of κ found with the precise version, 1.5433, is considerable lower than the value found with the Schwartz approximation, 1.8855. Moreover, the value of λ found with the precise version is also lower than the value found with the Schwartz approximation (0.2181 and 0.2558 respectively). Finally, the value of σ_2 obtained with the precise and approximate versions is 0.3967 and 0.4622 respectively. In general looking at the Table we can appreciate that all the values found with the approximate version used by Schwartz (1997) are higher than the corresponding values found with the precise version. Therefore, we can conclude that, at least with this data set, the approximate version by Schwartz (1997) tends to overestimate the parameters.

Figures 2 and 3 present the differences between one month WTI futures prices and the spot price calculated with both the precise and the approximated estimates[10]. Specifically, Figure 2 compares the predictive ability of both estimates in terms of the

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[9] This is one of the data sets used in Schwartz (1997). However in that paper the data set includes 510 observations, instead of 529. That is the reason why the results presented here for Schwartz approximation are not exactly the same as the ones presented in Schwartz (1997).

[10] To the best of our knowledge, there is no reliable index which reflects the WTI crude oil spot price. Therefore, the best available approximation for it, NYMEX WTI crude oil futures contracts with one month to maturity, is used.

mean error (ME), defined as the average of the series of one month futures price minus estimated spot prices, whereas in Figure 3 it is used the root mean squared error (RMSE).

In the full sample period, 1985-1995, the precise estimates outperform the approximation by Schwartz (1997), using the two metrics. This is also the case in all the annual periods considered in the Figures. However, it is interesting to note that the best performance of the precise estimates is found in 1985 and 1990, years which are characterized by high volatility, as can be appreciated in Figure 1. This fact is not surprising since, as pointed out above, one of the main advantage of the precise methodology is that it provides a more accurate estimation of the variance of the residual, $Var[\psi_t]$, which is given by expression (10). Finally, it is worth noting that the mean error is negative in the whole sample period, implying that both estimates tend to overestimate spot prices. It is also the case in all the annual periods, except for 1986, 1993 and 1994.

Figures 4 and 5 show the differences between one month WTI futures and spot prices calculated with both the precise and the approximated estimates, by month. The results are similar to those obtained in Figures 2 and 3, i.e. the precise estimates outperform the approximation by Schwartz (1997), using the two metrics (mean error and root mean squared error), in all months, except for March with the mean error measure.

Finally, Table 4 compares the improvement[11] (expressed in percentage) in the RMSE and the standard deviation of one-month futures price, by month. Interestingly, the highest improvement in the RMSE is obtained in October and November, which are that the months characterized by the highest degree of variance. As pointed out above, this result can be related with the fact that one of the main advantages of the precise estimation procedure is that it provides a more accurate estimation of the variance of the residual, $Var[\psi_t]$, which is given by expression (10). It should be noted, however, that there are also months with no such high variance showing a high improvement in the RMSE (January and December).

4. The simplified deduction of the futures prices in the two-factor model by Schwartz and Smith (2000)

Let us consider the two-factor model in Schwartz and Smith (2000). They assume that the spot log-price of a commodity at time *t*, ln(S_t), can be decomposed as the sum of a short-term deviation, χ_t, and the equilibrium price level, ξ_t: $\ln(S_t) = \chi_t + \xi_t$.

The short-term deviation and the equilibrium level are assumed to follow a mean-reverting process (toward zero) and a standard Brownian motion respectively, i.e.:

$$\begin{cases} d\chi_t = -\kappa\chi_t dt + \sigma_\chi d\, z_\chi \\ d\xi_t = \mu_\xi dt + \sigma_\xi dz_\xi \end{cases}$$

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[11] Defined as the RMSE computed with the Schwartz approximation minus the RMSE computed with the precise version of the estimates.

Where dz_χ and dz_ξ are standard Brownian motions with correlation ρ, i.e. $dz_\chi dz_\xi = \rho\, dt$, κ represents the rate at which the short-term deviations revert toward zero (the mean-reverting coefficient), μ_ξ is the equilibrium total return and σ_χ and σ_ξ are the volatilities of the short-term deviation and the equilibrium level respectively.

The risk-neutral version of their model is given by the following SDE:

$$\begin{cases} d\chi_t = (-\kappa\chi_t - \lambda_\chi)dt + \sigma_\chi d z_\chi^* \\ d\xi_t = \mu_\xi^* dt + \sigma_\xi dz_\xi^* \end{cases}$$

Where dz_χ^* and dz_ξ^* are again standard Brownian motions with correlation ρ, i.e. $dz_\chi dz_\xi = \rho\, dt$, $\mu_\xi^* = \mu_\xi - \lambda_\xi$, and λ_χ and λ_ξ are the market prices of risk associated to the short-term deviation and the equilibrium level respectively.

Defining the state vector as $X_t = (\chi_t, \xi_t)'$, the model can be expressed as[12]:

$$dX_t = \left[\begin{pmatrix} -\lambda_\chi \\ \mu_\xi^* \end{pmatrix} + \begin{pmatrix} -\kappa & 0 \\ 0 & 0 \end{pmatrix} X_t\right] dt + R dW_t$$

where *R* is the Choleski decomposition of the noise covariance matrix[13]:

$$\begin{pmatrix} \sigma_\chi^2 & \rho\sigma_\chi\sigma_\xi \\ \rho\sigma_\chi\sigma_\xi & \sigma_\xi^2 \end{pmatrix}$$

Now, we will use expressions (3) and (4). Note that, as A is diagonal, $P = I$ so we can safely drop P and P^{-1} from all expressions.

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[12] See Appendix B.

[13] Note again that R does not need to be calculated as RR' is the noise covariance matrix.

It is easy to see that (note that, in order to comply with Schwartz and Smith's notation, $D = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix}$, the null part is in the bottom of the matrix):

$$J(t) = \begin{pmatrix} \dfrac{1-e^{-\kappa t}}{\kappa} & 0 \\ 0 & t \end{pmatrix} \quad \exp(At) = \begin{pmatrix} e^{-\kappa t} & 0 \\ 0 & 1 \end{pmatrix}$$

$$G(t) = \begin{pmatrix} e^{-\kappa t} & 0 \\ 0 & 1 \end{pmatrix} vec^{-1} \left[\begin{pmatrix} \dfrac{e^{2\kappa t}-1}{2\kappa} & 0 & 0 & 0 \\ 0 & \dfrac{e^{\kappa t}-1}{\kappa} & 0 & 0 \\ 0 & 0 & \dfrac{e^{\kappa t}-1}{\kappa} & 0 \\ 0 & 0 & 0 & t \end{pmatrix} \begin{pmatrix} \sigma_\chi^2 \\ \rho\sigma_\chi\sigma_\xi \\ \rho\sigma_\chi\sigma_\xi \\ \sigma_\xi^2 \end{pmatrix} \right] \begin{pmatrix} e^{-\kappa t} & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-\kappa t} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \dfrac{e^{2\kappa t}-1}{2\kappa}\sigma_\chi^2 & \dfrac{e^{\kappa t}-1}{\kappa}\rho\sigma_\chi\sigma_\xi \\ \dfrac{e^{\kappa t}-1}{\kappa}\rho\sigma_\chi\sigma_\xi & t\sigma_\xi^2 \end{pmatrix} \begin{pmatrix} e^{-\kappa t} & 0 \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \dfrac{1-e^{2\kappa t}}{2\kappa}\sigma_\chi^2 & \dfrac{1-e^{\kappa t}}{\kappa}\rho\sigma_\chi\sigma_\xi \\ \dfrac{1-e^{\kappa t}}{\kappa}\rho\sigma_\chi\sigma_\xi & t\sigma_\xi^2 \end{pmatrix}$$

Now, the mean and variance of X_t are:

$$E^*[X_t] = \begin{pmatrix} -(1-e^{-kt})\lambda_\chi / k \\ \mu_\xi^* \end{pmatrix} + \begin{pmatrix} e^{-\kappa t} & 0 \\ 0 & 1 \end{pmatrix} X_0$$

$$Var^*[X_t] = G(t) = \begin{pmatrix} \left(1-e^{-2\kappa t}\right)\sigma_\chi^2 / 2\kappa & \left(1-e^{-\kappa t}\right)\rho\sigma_\chi\sigma_\xi / \kappa \\ \left(1-e^{-\kappa t}\right)\rho\sigma_\chi\sigma_\xi / \kappa & \sigma_\xi^2 t \end{pmatrix}$$

In this model, the log of spot price, Y_t = ln(S_t), is given by $\chi_t + \xi_t$. Thus, ln(S_t) is a Gaussian variable with mean:

$$e^{-\kappa t}\chi_0 + \xi_0 + \mu_\xi^* t - (1-e^{-kt})\lambda_\chi / k$$

and variance:

$$\left(1-e^{-2\kappa t}\right)\sigma_{\chi}^{2}/2\kappa+2\left(1-e^{-\kappa t}\right)\rho\sigma_{\chi}\sigma_{\xi}/\kappa+\sigma_{\xi}^{2}t\,.$$

Finally, the spot price, S_t, is lognormal distributed, and, therefore, the futures price can be written as:

$$F_{0,T}=E^{*}\left[S_{T}\right]=\exp\left(E^{*}\left[Y_{T}\right]+\frac{1}{2}Var^{*}\left[Y_{T}\right]\right)=$$
$$=\exp\left\{e^{-\kappa t}\chi_{0}+\xi_{0}+\mu_{\xi}^{*}t-(1-e^{-kt})\lambda_{\chi}/k+\frac{\left(1-e^{-2\kappa t}\right)\sigma_{\chi_{\chi}}^{2}/2\kappa+2\left(1-e^{-\kappa t}\right)\rho\sigma_{\chi}\sigma_{\xi}/\kappa+\sigma_{\xi}^{2}t}{2}\right\}$$

We have obtained the same result as in Schwarz and Smith (2000), Equation 9, but in a simpler way, avoiding unnecessary limit steps.

5. Conclusions

The stochastic behaviour of commodity prices has been a common topic of research during the last years. However, the application of the standard Black-Scholes analysis is not straightforward, due to the complex dynamics of commodity prices. This is the reason why most of these studies present ad-hoc solutions, which are very complex and sometimes include approximations.

This article shows how to simplify formulae and deductions, and even compute an explicit matrix general formula, using well known techniques and results in stochastic calculus. This formula has been tested on real data and is a real alternative to programming each model separately.

Concretely, we show how to obtain more precise estimators of the parameters in the Schwartz (1997) two-factor model context, than the approximations given by the author. It is found that, in general, the approximations by Schwartz tend to overestimate the parameters. These divergences are important in the valuation of commodity contingent claims. Moreover, we have shown how to obtain the expression for the futures price given by Schwartz and Smith (2000) in a simpler way, avoiding unnecessary limit steps.

Appendix A: Mathematical reference results

In order to understand the results, it is necessary to introduce some mathematical preliminaries. All the concepts and formulae here shall be presented in an intuitive way, stressing the practical implementation.

First of all, we remind the reader some well known concepts. For an extensive review of matrix algebra and matrix derivatives, we recommend Magnus and Neudecker (1999).

- The derivative and integral of a time-dependent matrix (which we shall denote $A(t)$ or A_t indistinctly) are given element by element:

$$\frac{d}{dt}A(t)=\begin{pmatrix} \frac{d}{dt}a_{11}(t) & \ldots & \frac{d}{dt}a_{1n}(t) \\ \ldots & \ldots & \ldots \\ \frac{d}{dt}a_{m1}(t) & \ldots & \frac{d}{dt}a_{mn}(t) \end{pmatrix}, \qquad \int_r^s A(t)dt=\begin{pmatrix} \int_r^s a_{11}(t)dt & \ldots & \int_r^s a_{1n}(t)dt \\ \ldots & \ldots & \ldots \\ \int_r^s a_{m1}(t)dt & \ldots & \int_r^s a_{mn}(t)dt \end{pmatrix}.$$

  Indefinite integrals $\int A_t dt$ are defined in the same way. Linear properties, such as $\frac{d}{dt}(BA_t)=B\frac{d}{dt}A_t$, are easy to prove and shall be used without explicitly mentioning them.

- The matrix exponential of a diagonalizable matrix $A=PDP^{-1}$ with D diagonal is: $\exp(A)=P\begin{pmatrix} \exp(d_1) & 0 & 0 \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \exp(d_n) \end{pmatrix}P^{-1}$. It is not hard to see the equality

$$\frac{d}{dt}\exp(At)=A\exp(At)$$

- Given two matrices $A \in \Re^{pxq}, B \in \Re^{mxn}$ their Kronecker product is a *pm* x *qn* matrix defined as: $A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1q}B \\ a_{21}B & a_{22}B & \dots & a_{2q}B \\ \dots & \dots & \dots & \dots \\ a_{p1}B & a_{p2}B & \dots & a_{pq}B \end{pmatrix}$.

- The *vec* operator is defined as: $vec\begin{pmatrix} a_{11} & \dots & a_{1q} \\ \dots & \dots & \dots \\ a_{p1} & \dots & a_{pq} \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{p1} \\ a_{12} \\ \dots \\ a_{p2} \\ \dots \end{pmatrix}$.

- Integrals with a single product: We shall calculate $\int_r^s \exp(At)H\,dt$ where *H* is an arbitrary constant matrix. Let $A = PDP^{-1} = P\begin{pmatrix} 0 & 0 \\ 0 & D_1 \end{pmatrix}P^{-1}$ with D diagonal and D_1 non-singular. The previous integral is therefore easily computed explicitly as:

$$\int_r^s \exp(At)Hdt = P\left(\int_r^s \exp(Dt)dt\right)P^{-1}H = \left[P\begin{pmatrix} tI & 0 \\ 0 & \exp(D_1 t) \end{pmatrix}P^{-1}H\right]_r^s =$$
$$P\begin{pmatrix} (s-r)I & 0 \\ 0 & D_1^{-1}(\exp(D_1 s) - \exp(D_1 r)) \end{pmatrix}P^{-1}H$$

- Integrals with double product: We shall calculate $\int_r^s U\exp(At)H\exp(At)^{'}V\,dt$, where *U, H, V* are arbitrary constant matrices. As before: $A = PDP^{-1} = P\begin{pmatrix} 0 & 0 \\ 0 & D_1 \end{pmatrix}P^{-1}$,

$\int_r^s U\exp(At)H\exp(At)^{'}V\,dt = UP\left(\int_r^s \exp(Dt)P^{-1}H(P^{-1})^{'}\exp(Dt)^{'}\,dt\right)P^{'}V$ so we shall focus on the middle part. Using the vec operator:

$$\int_r^s \exp(Dt) H \exp(Dt)' \, dt = vec^{-1}\left[vec\left(\int_r^s \exp(Dt)\left(P^{-1} H \left(P^{-1}\right)'\right) \exp(Dt)' \, dt \right)\right] =$$

$$vec^{-1}\left[\int_r^s \left(\exp(Dt) \otimes \exp(Dt)\right) vec\left(P^{-1} H \left(P^{-1}\right)'\right) dt \right] \; = vec^{-1}\left\{ \left[\int_r^s \exp(Dt) \otimes \exp(Dt) dt \right] vec\left(P^{-1} H \left(P^{-1}\right)'\right)$$

The only thing left is to compute the central integral. However, if *D* is diagonal,

let $D = \begin{pmatrix} 0 & \dots & \dots & 0 \\ 0 & d_1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & d_n \end{pmatrix}$. Then $\exp(Dt) = \begin{pmatrix} 0 & \dots & \dots & 0 \\ 0 & e^{d_1 t} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & e^{d_n t} \end{pmatrix}$. The Kronecker

product is thus given by: $\exp(Dt) \otimes \exp(Dt) = \begin{pmatrix} 0 & \dots & \dots & 0 \\ 0 & e^{(d_1 I + D_1)t} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & e^{(d_n + D_1)t} \end{pmatrix}$. If no

eigenvalue is exactly the opposite of another eigenvalue the integral is given by

$$\int_r^s \exp(Dt) \otimes \exp(Dt) = \begin{pmatrix} (r-s)I & \dots & \dots & 0 \\ 0 & (d_1 I + D_1)^{-1} e^{(d_1 I + D)t} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & (d_k I + D_1)^{-1} e^{(d_n + D)t} \end{pmatrix} \text{I}$$

f two eingenvalues are one the opposite of the other, matters are not much

more difficult. Let $D = \begin{pmatrix} \mu_1 & 0 & \dots & 0 \\ 0 & \mu_2 & \dots & 0 \\ \dots & \dots & \dots & 0 \\ 0 & 0 & \dots & \mu_k \end{pmatrix}$ including all zero and nonzero

eigenvalues. If we just let $\gamma_{ij} = \mu_i + \mu_j$ and substitute in the formula, we have

$\exp(Dt) \otimes \exp(Dt) = \exp\left[\begin{pmatrix} \gamma_{11} & 0 & 0 & 0 & \dots & 0 \\ 0 & \gamma_{12} & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \gamma_{1k} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \dots & \gamma_{kk} \end{pmatrix} t \right]$ and its integral is:

$$\int_r^s \exp(Dt)\otimes\exp(Dt)=\begin{pmatrix} \int_r^s e^{\gamma_{11}t}dt & 0 & 0 & 0 & \dots & 0 \\ 0 & \int_r^s e^{\gamma_{12}t}dt & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \int_r^s e^{\gamma_{1k}t}dt & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \dots & \int_r^s e^{\gamma_{kk}t}dt \end{pmatrix}.$$

Where obviously $\int_r^s e^{\gamma_{ij}t}dt=\begin{cases} s-r & \text{for } \gamma_{ij}=0 \\ \dfrac{e^{\gamma_{ij}s}-e^{\gamma_{ij}r}}{\gamma_{ii}} & \text{for } \gamma_{ij}\neq 0 \end{cases}.$

- Note that the expression $vec^{-1}\left\{\left[\int_r^s \exp(Dt)\otimes\exp(Dt)dt\right]vec\left(P^{-1}H\left(P^{-1}\right)'\right)\right\}$ can be done in a different way, using the Hadamard product instead of the Kronecker one and thus avoiding the use of diagonal matrices. To do so, remember that the Hadamard product of A and B denoted $A\bullet B$ is defined each element at a time: $(A\bullet B)_{ij}=A_{ij}B_{ij}$. If we just define $Z=vec^{-1}\left(\int_r^s \exp(Dt)\otimes\exp(Dt)dt\right)$ or equivalently $Z_{ij}=\int_r^s e^{\gamma_{ij}t}dt$, then it is easy to notice, just by substitution, that $vec^{-1}\left\{\left[\int_r^s \exp(Dt)\otimes\exp(Dt)dt\right]vec\left(P^{-1}H\left(P^{-1}\right)'\right)\right\}$ equals $ZP^{-1}H\left(P^{-1}\right)'$. The reader should note, however, that due to the fact that our Kronecker product is diagonal, it does not have to be stored in full, so an efficient implementation of the algorithm will use only the diagonal

All operations are easily implemented in any mathematically adapted computer language.

Appendix B: Futures Contract Valuation

Most of the models proposed in the literature assume that the risk-neutral dynamics of a commodity price (or its log) is given by a linear stochastic differential system:

$$\begin{cases} dX_t = (b + AX_t)dt + RdW_t \\ Y_t \quad = cX_t \end{cases}$$

where Y_t is the commodity price (or its log), *b*, *A*, *R* and *c* are deterministic parameters[14] independent of *t* ($b \in \Re^n, A, R \in \Re^{n\,x\,n}$, $c \in \Re^n$) and W_t is a *n*-dimensional canonical Brownian motion (i.e. all components uncorrelated and its variance equal to unity) under the risk-neutral measure.

Let us see that the solution of that problem is[15]:

$$X_t = e^{At}\left[X_0 + \int_0^t e^{-As} bds + \int_0^t e^{-As} RdW_s\right] \tag{B1}$$

In order to proof it, we shall apply the general rule for the derivation of the product of stochastic components (Oksendal, 1992):

$$dX_t = \left(de^{At}\right)\left[X_0 + \int_0^t e^{-As} bds + \int_0^t e^{-As} RdW_s\right] + e^{At} d\left[X_0 + \int_0^t e^{-As} bds + \int_0^t e^{-As} RdW_s\right] + \\ + \left(de^{At}\right) d\left[X_0 + \int_0^t e^{-As} bds + \int_0^t e^{-As} RdW_s\right]$$

It is easy to show that:

$$d\left[X_0 + \int_0^t e^{-As} bds + \int_0^t e^{-As} RdW_s\right] = e^{-At} bdt + e^{-At} RdW_t$$

---

[14] Again note that *R* does not need to be computed.

[15] Even in the case that *b*, *A* and *R* were function of *t*, if A_t and $\int_0^t A_s ds$ commute, the solution of that problem is (B1).

The first differential only has elements of type *dt*, hence the product of the first differential times the second differential is zero.

Thus:

$$dX_t = Ae^{At}dt\left[X_0 + \int_0^t e^{-As}bds + \int_0^t e^{-As}RdW_s\right] + e^{At}\left[e^{-At}bdt + e^{-At}RdW_t\right] = A_t X_t dt + bdt + RdW_t$$

Consequently we obtain expression (B1):

$$X_t = e^{At}\left[X_0 + \int_0^t e^{-As}bds + \int_0^t e^{-As}RdW_s\right].$$

It is easy to prove that the solution is unique (Oksendal, 1992).

An elementary rule of the stochastic calculus states that if J_s is a deterministic function, $\int_0^t J_s dW_s$ is normally distributed with mean zero and variance:

$$Var\left(\int_0^t J_s dW_s\right) = \int_0^t J_s J_s^T ds \text{ (Itô´s isometry).}$$

Accordingly, X_t is normally distributed with mean and variance[16]:

$$E^*[X_t] = e^{At}\left[X_0 + \int_0^t e^{-As}\, bds\right] \tag{B2}$$

$$Var^*[X_t] = e^{At}\left[\int_0^t e^{-As}RR'e^{-As'}ds\right]e^{At'} \tag{B3}$$

Therefore, Y_t, under the risk-neutral measure, is also Gaussian and it easily follows that its mean and variance are: $E^*[Y_t] = cE^*[X_t]$, $Var^*[Y_t] = cVar^*[X_t]c'$, providing a valuation scheme for all sorts of commodity contingent claims as financial derivatives on commodity prices, real options, investment decisions and other more.

If Y_t is the log of the commodity price (S_t), the price of a futures contract traded at time *t* with maturity at time *t+T*, $F_{t,T}$, can be computed as:

---

[16] E*[] and Var*[] are the mean and variance under the risk neutral measure.

$$F_{t,T} = E^*\left[S_{t+T} \| I_t\right] = \exp\left\{E^*\left[Y_{t+T} \| I_t\right] + \frac{1}{2} Var^*\left[Y_{t+T} \| I_t\right]\right\} \qquad \text{(B4)}$$

where I_t is the information available at time t.

This methodology can be used in all kind of problems (even if *b*, *A* and *R* are functions of *t*, although, in this case the explicit formulae for the integrals, given in appendix A, do not apply). Moreover, this methodology is much simpler than the ad-hoc solutions presented in the literature that can only be used in the concrete problem for which they were developed and need complex procedures like limit steps (Schwartz and Smith, 2000) or partial differential equations (Schwartz, 1997).

Appendix C: Volatility of Futures Returns

The squared volatility of a futures contract traded at time *t* with maturity at time *t+T* is defined as[17]:

$$\lim_{h \to 0} \frac{Var\left[\log F_{t+h,T} - \log F_{t,T}\right]}{h}.$$

We will prove that it is the expected value of the square of the coefficient of the Brownian motion (σ_t) in the expression $d\log\left(F_{t,T}\right) = \mu_t ds + \sigma_t dW_t^F$, where W_t^F is a scalar canonical Brownian motion, as long as μ_t is mean squared bounded in an

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[17] The same results are going to be obtained if the volatility is defined as:
$$\lim_{h \to 0} \frac{Var\left[\log F_{t+h,T-h} - \log F_{t,T}\right]}{h}.$$

interval containing t (it does not matter whether it is a function of $F_{t,T}$ or not) and $E[\sigma_t^2]$ is continuous in t[18].

Expressing $d\log F_{t,T} = \mu_t dt + \sigma_t dW_t$ in the equivalent integral form:

$$\log F_{t+h,T} - \log F_{t,T} = \int_t^{t+h} \mu_s ds + \int_t^{t+h} \sigma_s dW_s \ ,$$

its expected value is $\int_t^{t+h} E[\mu_s]ds$. Therefore, its variance is given by:

$$Var[\log F_{t+h,T} - \log F_{t,T}] = E\left[\left(\int_t^{t+h} \mu_s - E[\mu_s]ds + \int_t^{t+h} \sigma_s dW_s\right)^2\right].$$

Using standard properties:

$$E\left[\left(\int_t^{t+h} \mu_s - E[\mu_s]ds + \int_t^{t+h} \sigma_s dW_s\right)^2\right] = E\left[\left(\int_t^{t+h} \mu_s - E[\mu_s]ds\right)^2\right] + E\left[\left(\int_t^{t+h} \sigma_s dW_s\right)^2\right]$$

as μ_t is non-anticipating.

By Itô's isometry: $E\left[\left(\int_t^{t+h} \sigma_s dW_s\right)^2\right] = \int_t^{t+h} E[\sigma_s^2]\,ds$

Taking limits and using the mean value theorem of the integral calculus:

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} E[\sigma_s^2]\,ds = E[\sigma_t^2].$$

For the other term it can be seen that:

$$E\left[\left(\int_t^{t+h} \mu_s - E[\mu_s]ds\right)^2\right] = \left\|\int_t^{t+h} \mu_s - E[\mu_s]ds\right\|_2^2 \le \left(\int_t^{t+h} \|\mu_s - E[\mu_s]\|_2 ds\right)^2$$

As for some δ >0, μ_t is mean squared bounded in the interval (t-δ, t+δ), when $h \to 0$, this integral is less or equal than $h^2 \sup\{\|\mu_s - E[\mu_s]\|_2 : s \in (t-\delta, t+\delta)\}$, and $\sup\{\|\mu_s - E[\mu_s]\|_2 : s \in (t-\delta, t+\delta)\} \le M$ for some *M*. Hence,

---

[18] In the general problem of this article these conditions are satisfied.

$$\frac{1}{h}E\left[\left(\int_t^{t+h} \quad \mu_s - E[\mu_s]ds\right)^2\right] \leq \frac{1}{h}h^2 M$$

which converges to 0 when $h \to 0$.

Therefore:

$$\lim_{h \to 0} \frac{Var\left[\log F_{t+h,T} - \log F_{t,T}\right]}{h} = E\left[\sigma_t^2\right].$$

Hence, taking logarithms and differentials on both sides of Equation (B4), it follows that:

$$d\left(\log F_{t,T}\right) = ce^{AT} dX_t = ce^{AT}\left[b + AX_t\right]dt + ce^{AT} R\, dW_t$$

Therefore, the squared volatility is[19]:

$$ce^{AT} RR' e^{AT}{}' c'.$$

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[19] Again note that R needs not to be computed as RR' is the noise covariance matrix.

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TABLE 1

TIME (MILISECONDS) NEEDED FOR AN EVALUATION OF THE LOG-LIKELIHOOD FUNCTION

Integral stands for using a symbolic processor to compute the integral each step. General means using the same script (formulae (3) and (4) in matrix form) for all models and Particular means writing down the formulae for each case.

Data	Brent		Heating oil		WTI	
Factors	2	3	2	3	2	3
Integral	2785.00	7881.34	3316.16	14774.04	5404.36	3916.64
General	61.48	64.28	55.48	56.08	75.52	89.12
Particular	47.08	49.88	33.06	34.64	57.48	70.10

TABLE 2

TIME (SECONDS) FOR A FULL ESTIMATION OF A MODEL

General means using the same script (formulae (3) and (4) in matrix form) for all models and Particular means writing down the formulae for each case. Integrating symbolically each step would be computationally burdensome.

Data	Brent		Heating oil		WTI	
Factors	2	3	2	3	2	3
General	74.10	250.02	59.39	180.23	91.70	210.33
Particular	60.26	220.97	39.31	128.06	69.53	234.42

FIGURE 1

WTI FUTURES PRICE WITH ONE MONTH TO MATURITY

TABLE 3

THE TWO-FACTOR MODEL BY SCHWARTZ (1997). PRECISE AND APPROXIMATE ESTIMATES

The Table shows the parameter estimates obtained with the Schwartz (1997) approximation and with the precise method described in this paper. Standard errors in parenthesis.

Parameter	Precise Method	Schwartz Approximation
μ	0.1629 (0.0725)	0.1678 (0.0732)
k	1.5433 (0.0318)	1.8855 (0.0356)
α	0.1458 (0.0558)	0.1496 (0.0545)
σ_1	0.3278 (0.0073)	0.3293 (0.0072)
σ_2	0.3967 (0.0113)	0.4622 (0.0119)
ρ	0.8073 (0.0104)	0.8084 (0.0107)
λ	0.2181 (0.0864)	0.2558 (0.1029)

FIGURE 2

MEAN ERROR BY YEAR

The Figure shows the differences (mean error) between the one month futures price and the spot price calculated with precise and approximated estimates, by year.

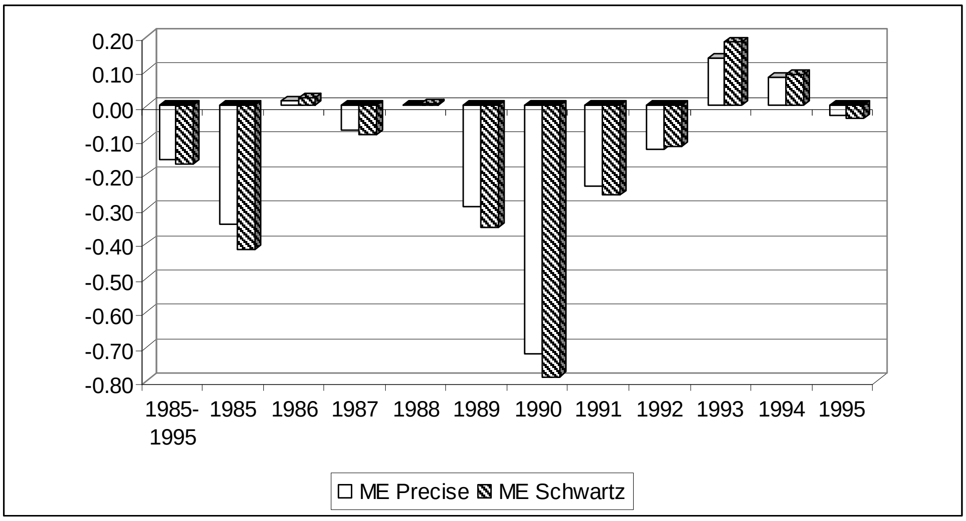

FIGURE 3

ROOT MEAN SQUARED ERROR BY YEAR

The Figure shows the differences (root mean squared error) between the one month futures price and the spot price calculated with precise and approximated estimates, by year.

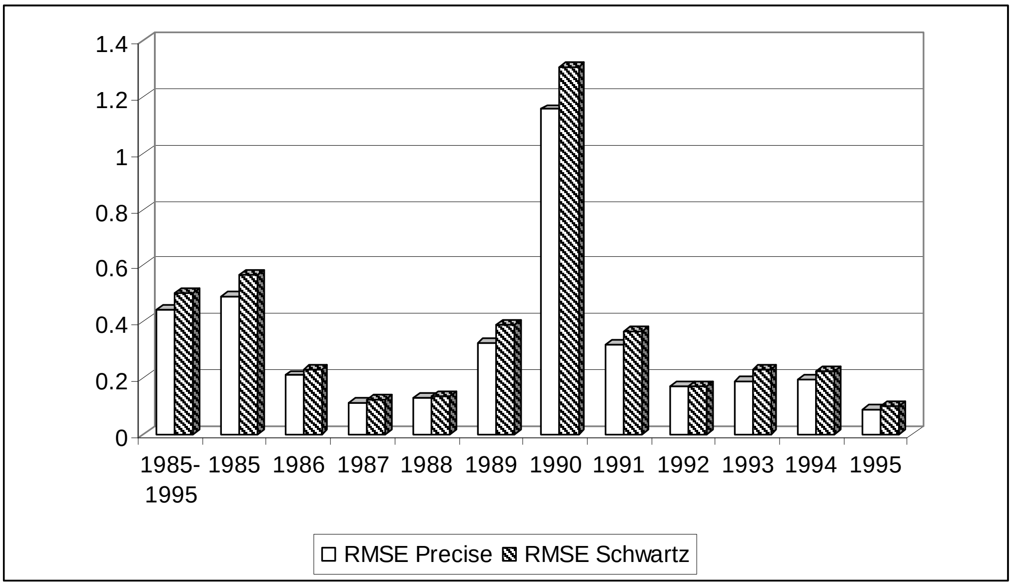

FIGURE 4

MEAN ERROR BY MONTH

The Figure shows the differences (mean error) between the one month futures price and the spot price calculated with both precise and approximated estimates, by month.

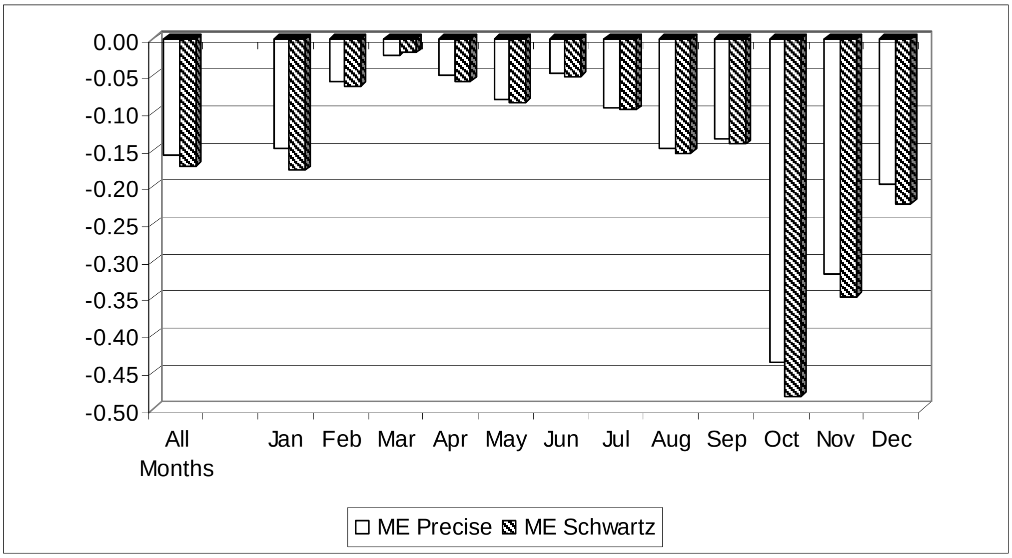

FIGURE 5

ROOT MEAN SQUARED ERROR BY MONTH

The Figure shows the differences (root mean squared error) between the one month futures price and the spot price calculated with both precise and approximated estimates, by month.

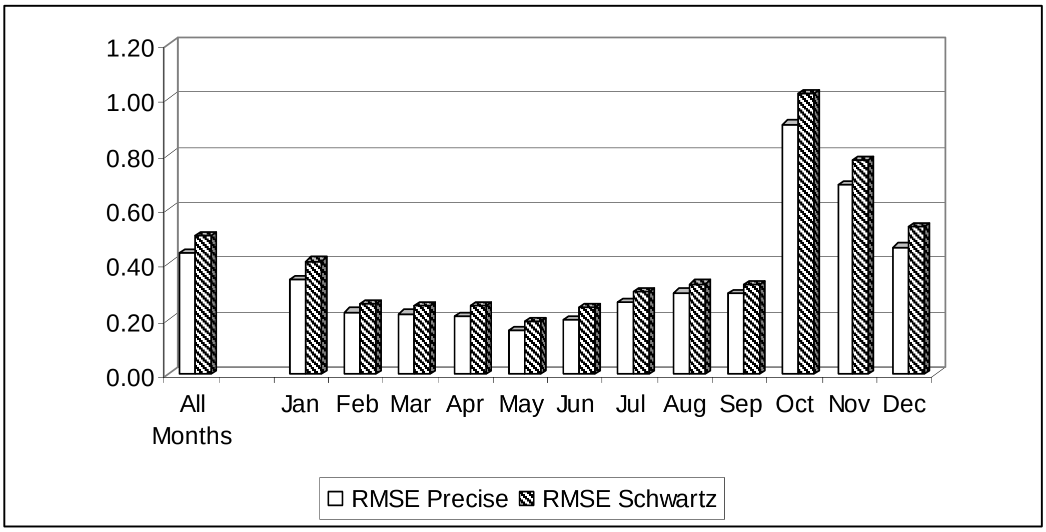

TABLE 4

COMPARISON OF THE IMPROVEMENT IN THE RMSE AND ONE-MONTH FUTURES PRICE STANDAR DEVIATION BY MONTH

The Table shows the improvement (expressed in percentage) in the RMSE, defined as the RMSE computed with the Schwartz approximation minus the RMSE computed with the precise version of the estimates, and one-month futures price standard deviation, by month.

	Improvement RMSE (%)	Volatility
All Months	6.06341562	4.5066963
January	6.69700526	3.45920263
February	2.90069147	3.43375304
March	2.86456161	3.9271667
Aprril	3.82981177	3.88082312
May	3.20130602	3.37948674
June	4.20386706	3.61776438
July	4.02239618	4.05271984
August	3.25451898	4.14305907
September	3.37241986	4.25738991
October	11.0467666	6.73405967
November	8.73584998	5.73730612
December	7.36128089	4.18504435